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Sharp large deviations for the non-stationary Ornstein–Uhlenbeck process

Bernard Bercu[a,*], Laure Coutin[b], Nicolas Savy[b]

[a] Université Bordeaux 1, Institut de Mathématiques de Bordeaux, UMR C5251, 351 cours de la libération, 33405 Talence Cedex, France

[b] Université Paul Sabatier, Institut de Mathématiques de Toulouse, UMR C5583, 31062 Toulouse Cedex 09, France

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Abstract

For the Ornstein–Uhlenbeck process, the asymptotic behavior of the maximum likelihood estimator of the drift parameter is totally different in the stable, unstable, and explosive cases. Notwithstanding this trichotomy, we investigate sharp large deviation principles for this estimator in the three situations. In the explosive case, we exhibit a very unusual rate function with a shaped flat valley and an abrupt discontinuity point at its minimum.

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1. Introduction

Consider the Ornstein–Uhlenbeck process observed over the time interval $[0, T]$

$$dX_t = \theta X_t dt + dB_t \tag{1.1}$$

where (B_t) is a standard Brownian motion and the drift θ is an unknown real parameter. For the sake of simplicity, we choose the initial state $X_0 = 0$. The process is said to be stable if $\theta < 0$,

* Corresponding author.

E-mail addresses: Bernard.Bercu@math.u-bordeaux1.fr (B. Bercu), laure.coutin@math.univ-toulouse.fr (L. Coutin), nicolas.savy@math.univ-toulouse.fr (N. Savy).

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unstable if $\theta = 0$, and explosive if $\theta > 0$. The maximum likelihood estimator of θ is given by

$$\widehat{\theta}_T = \frac{\int_0^T X_t dX_t}{\int_0^T X_t^2 dt} = \frac{X_T^2 - T}{2\int_0^T X_t^2 dt}. \tag{1.2}$$

It is well-known (see e.g. [15, p. 234]) that in the stable, unstable, and explosive cases

$$\lim_{T\to\infty} \widehat{\theta}_T = \theta \quad \text{a.s.}$$

However, the asymptotic normality is totally different in the three situations. As a matter of fact, if $\theta < 0$, the process (X_T) is positive recurrent and Brown and Hewitt [5] have shown the asymptotic normality

$$\sqrt{T}(\widehat{\theta}_T - \theta) \xrightarrow{\mathcal{L}} \mathcal{N}(0, -2\theta).$$

Moreover, if $\theta = 0$, the process (X_T) is null recurrent and it was proved by Feigin [10] that

$$T(\widehat{\theta}_T - \theta) \xrightarrow{\mathcal{L}} \frac{\int_0^1 B_t dB_t}{\int_0^1 B_t^2 dt} = \frac{B_1^2 - 1}{2\int_0^1 B_t^2 dt}$$

where (B_t) is a standard Brownian motion. Furthermore, if $\theta > 0$, the process (X_T) is transient and we know from [9] (see also [13, p. 304]) that

$$\exp(\theta T)(\widehat{\theta}_T - \theta) \xrightarrow{\mathcal{L}} 2\theta \left(\frac{Y}{Z}\right)$$

where Y, Z are two independent Gaussian $\mathcal{N}(0, 1)$ random variables which implies that the limiting ratio Y/Z has a Cauchy distribution. More recent contributions on parameter estimation for explosive Ornstein–Uhlenbeck processes may be found in [8,12,16]. Notwithstanding this trichotomy, our goal is to establish the large deviation properties for $(\widehat{\theta}_T)$ in the stable, unstable, and explosives cases. We refer the reader to the excellent book by Dembo and Zeitouni [7] on the theory of large deviations. First of all, in the stable case, Florens-Landais and Pham [11] proved the following large deviation principle (LDP) for $(\widehat{\theta}_T)$.

Lemma 1.1. *If $\theta < 0$, then $(\widehat{\theta}_T)$ satisfies an LDP with speed T and good rate function*

$$I(c) = \begin{cases} -\dfrac{(c-\theta)^2}{4c} & \text{if } c < \dfrac{\theta}{3}, \\ 2c - \theta & \text{otherwise.} \end{cases} \tag{1.3}$$

This result was extended by the following sharp large deviation principle (SLDP) for $(\widehat{\theta}_T)$ established by Bercu and Rouault [4].

Theorem 1.2. *Consider the Ornstein–Uhlenbeck process given by (1.1) where the drift parameter $\theta < 0$.*

(a) *For all $c < \theta$, there exists a sequence $(b_{c,k})$ such that, for any $p > 0$ and T large enough,*

$$\mathbb{P}(\widehat{\theta}_T \le c) = \frac{-\exp(-TI(c) + H(c))}{a_c \sigma_c \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{b_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \tag{1.4}$$

where

$$a_c = \frac{c^2 - \theta^2}{2c} \quad \textit{and} \quad \sigma_c^2 = -\frac{1}{2c} \tag{1.5}$$

$$H(c) = -\frac{1}{2}\log\left(\frac{(c+\theta)(3c-\theta)}{4c^2}\right) \tag{1.6}$$

while, for all $\theta < c < \theta/3$,

$$\mathbb{P}(\widehat{\theta}_T \geq c) = \frac{\exp(-TI(c) + H(c))}{a_c \sigma_c \sqrt{2\pi T}}\left[1 + \sum_{k=1}^{p} \frac{b_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]. \tag{1.7}$$

(b) *For all* $c > \theta/3$ *with* $c \neq 0$, *there exists a sequence* $(d_{c,k})$ *such that, for any* $p > 0$ *and* T *large enough,*

$$\mathbb{P}(\widehat{\theta}_T \geq c) = \frac{\exp(-TI(c) + K(c))}{a_c \sigma_c \sqrt{2\pi T}}\left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \tag{1.8}$$

where

$$a_c = 2(c-\theta) \quad \textit{and} \quad \sigma_c^2 = \frac{c^2}{2(2c-\theta)^3} \tag{1.9}$$

$$K(c) = -\frac{1}{2}\log\left(\frac{(c-\theta)(3c-\theta)}{4c^2}\right). \tag{1.10}$$

(c) *For* $c = \theta/3$, *there exists a sequence* (e_k) *such that, for any* $p > 0$ *and* T *large enough,*

$$\mathbb{P}(\widehat{\theta}_T \geq c) = \frac{exp(-TI(c))}{2\pi T^{1/4}} \frac{\Gamma(1/4)}{a_\theta^{3/4}\sigma_\theta}\left[1 + \sum_{k=1}^{2p} \frac{e_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right] \tag{1.11}$$

where

$$a_\theta = -\frac{4\theta}{3} \quad \textit{and} \quad \sigma_\theta^2 = -\frac{3}{2\theta}. \tag{1.12}$$

(d) *Finally, for* $c = 0$, $p > 0$ *and for* T *large enough,*

$$\mathbb{P}(\widehat{\theta}_T \geq 0) = 2\frac{\exp(-TI(c))}{\sqrt{2\pi T}\sqrt{-2\theta}}\left[1 + \sum_{k=1}^{p} \frac{(2k)!}{2^{2k}\theta^k T^k k!} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]. \tag{1.13}$$

Our purpose is to extend this investigation by establishing SLDP for $(\widehat{\theta}_T)$ in the explosive and unstable cases. Similar results in discrete time for the Gaussian autoregressive process may be found in [1]. We also refer the reader to [2] where SLDP for the maximum likelihood estimator of θ is proved for the stable Ornstein–Uhlenbeck process driven by a fractional Brownian motion. We wish to mention here that it should be possible to extend the previous work of Zani [17] to generalized squared radial Ornstein–Uhlenbeck processes with parameter $\theta > 0$, given by

$$dX_t = 2\theta X_t dt + \delta dt + 2\sqrt{X_t}dB_t.$$

As a matter of fact, as soon as the dimensional parameter $\delta > 0$ is known, the maximum likelihood estimator of θ is

$$\widetilde{\theta}_T = \frac{X_T - \delta T}{2\int_0^T X_t dt}.$$

This estimator is quite similar to the maximum likelihood estimator $\widehat{\theta}_T$ given by (1.2). Consequently, it should be possible to establish SLDP for this estimator in the explosive case $\theta > 0$. Finally, SLDP are very useful in practical situations as the numerical approximations calculated via SLDP outperform those obtained with the central limit theorem or with Edgeworth expansions in a wide range of statistical applications [3].

The paper is organized as follows. In Section 2, we propose a keystone lemma which is at the core of all our analysis. Section 3 is devoted to the main results of the paper while Section 4 contains their proofs. All the technical proofs of Sections 2 and 4 are postponed to Appendices A–D.

2. A keystone lemma

The sharp large deviations properties of $(\widehat{\theta}_T)$ are closely related to those of

$$Z_T(c) = \int_0^T X_t dX_t - c\int_0^T X_t^2 dt$$

with $c \in \mathbb{R}$ since $\mathbb{P}(\widehat{\theta}_T \geq c) = \mathbb{P}(Z_T(c) \geq 0)$. One has to keep in mind that the threshold c for $\widehat{\theta}_T$ appears like a parameter for Z_T. Denote by $\mathcal{L}_T$ the normalized cumulant generating function of $Z_T(c)$

$$\mathcal{L}_T(a) = \frac{1}{T}\log \mathbb{E}\left[\exp(aZ_T(c))\right]$$

where the parameter c is omitted in order to simplify the notation. Moreover, let $\mathcal{L}$ be the pointwise limit of $\mathcal{L}_T$

$$\mathcal{L}(a) = -\frac{1}{2}\left(a + \theta + \sqrt{\theta^2 + 2ac}\right). \tag{2.1}$$

All our analysis relies on the following keystone lemma which is true as soon as the drift parameter $\theta \geq 0$.

Lemma 2.1. *Let $\Delta_c = \{a \in \mathbb{R},\ \theta^2 + 2ac > 0,\ a + \theta < \sqrt{\theta^2 + 2ac}\}$ be the effective domain of $\mathcal{L}$, that is the set of points in $\mathbb{R}$ for which $\mathcal{L}$ is finite, and set $\varphi(a) = -\sqrt{\theta^2 + 2ac}$, $\tau(a) = a + \theta - \varphi(a)$ and $h(a) = (a + \theta)/\varphi(a)$.*

(a) *For all $a \in \Delta_c$, we have*

$$\mathcal{L}_T(a) = \mathcal{L}(a) + \frac{1}{T}\mathcal{H}(a) + \frac{1}{T}\mathcal{R}_T(a) \tag{2.2}$$

where

$$\mathcal{H}(a) = -\frac{1}{2}\log\left(\frac{1}{2}(1 + h(a))\right), \tag{2.3}$$

$$\mathcal{R}_T(a) = -\frac{1}{2}\log\left(1 + \frac{1 - h(a)}{1 + h(a)}\exp(2\varphi(a)T)\right). \tag{2.4}$$

(b) *Moreover, the remainder $\mathcal{R}_T(a)$ goes to zero exponentially fast as*

$$\mathcal{R}_T(a) = \mathcal{O}\left(\exp(2\varphi(a)T)\right). \tag{2.5}$$

Proof. The proof of Lemma 2.1 is given in Appendix A. $\square$

3. Sharp large deviations results

We shall now focus our attention on the explosive case $\theta > 0$. It immediately follows from (1.1) that

$$X_T = \exp(\theta T) \int_0^T \exp(-\theta t) dB_t \tag{3.1}$$

leading to $\exp(-\theta T) X_T = Y_T$ where

$$Y_T = \int_0^T \exp(-\theta t) dB_t.$$

The Gaussian process (Y_T) converges almost surely and in mean square to the nondegenerate Gaussian random variable

$$Y = \int_0^\infty \exp(-\theta t) dB_t.$$

Hence, via Toeplitz's lemma,

$$\lim_{T \to \infty} \frac{1}{\exp(2\theta T)} \int_0^T X_t^2 dt = \frac{Y^2}{2\theta} \quad \text{a.s.}$$

Consequently, one can expect for $(\widehat{\theta}_T)$ an LDP with speed $\exp(2\theta T)$. However, $(\widehat{\theta}_T)$ is a sequence of self-normalized random variables and we shall now show that $(\widehat{\theta}_T)$ satisfies an LDP similar to that of Lemma 1.1 with speed T.

Lemma 3.1. *If $\theta > 0$, then $(\widehat{\theta}_T)$ satisfies an LDP with speed T and good rate function*

$$I(c) = \begin{cases} -\dfrac{(c-\theta)^2}{4c} & \text{if } c \le -\theta, \\ \theta & \text{if } |c| < \theta, \\ 0 & \text{if } c = \theta, \\ 2c - \theta & \text{if } c > \theta. \end{cases} \tag{3.2}$$

Remark 3.2. As for the Gaussian autoregressive process [1], one can observe that the rate function I in the explosive case is really unusual with a shaped flat valley and an abrupt discontinuity point at its minimum. It is possible to give some intuition on the size of the discontinuity jump. As a matter of fact, we already saw in the introduction that, if $\theta > 0$,

$$\exp(\theta T)(\widehat{\theta}_T - \theta) \xrightarrow{\mathcal{L}} 2\theta \left(\frac{Y}{Z}\right)$$

where Y, Z are two independent Gaussian $\mathcal{N}(0, 1)$ random variables. The size of the jump is precisely given by the logarithm of the rate $\exp(\theta T)$ properly normalized,

$$\frac{1}{T} \log(\exp(\theta T)) = \theta.$$

The SLDP for $(\widehat{\theta}_T)$, quite similar to the one established in the stable case, is as follows.

Theorem 3.3. *Consider the Ornstein–Uhlenbeck process given by (1.1) where the drift parameter $\theta > 0$.*

(a) *For all $c < -\theta$, there exists a sequence $(b_{c,k})$ such that, for any $p > 0$ and T large enough,*

$$\mathbb{P}(\widehat{\theta}_T \leq c) = \frac{-\exp(-TI(c) + H(c))}{a_c \sigma_c \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{b_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \tag{3.3}$$

where

$$a_c = \frac{c^2 - \theta^2}{2c} \quad \text{and} \quad \sigma_c^2 = -\frac{1}{2c} \tag{3.4}$$

$$H(c) = -\frac{1}{2} \log\left(\frac{(c+\theta)(3c-\theta)}{4c^2}\right). \tag{3.5}$$

(b) *For all $c > \theta$, there exists a sequence $(d_{c,k})$ such that, for any $p > 0$ and T large enough,*

$$\mathbb{P}(\widehat{\theta}_T \geq c) = \frac{\exp(-TI(c) + K(c))}{a_c \sigma_c \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \tag{3.6}$$

where

$$a_c = 2(c - \theta) \quad \text{and} \quad \sigma_c^2 = \frac{c^2}{2(2c-\theta)^3} \tag{3.7}$$

$$K(c) = -\frac{1}{2} \log\left(\frac{(c-\theta)(3c-\theta)}{4c^2}\right). \tag{3.8}$$

(c) *For all $|c| < \theta$ with $c \neq 0$, there exists a sequence $(e_{c,k})$ such that, for any $p > 0$ and T large enough,*

$$\mathbb{P}(\widehat{\theta}_T \leq c) = \frac{\exp(-TI(c) + J(c))}{a_c \sigma_c \sqrt{2\pi T}} \left[1 + \sum_{k=1}^{p} \frac{e_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \tag{3.9}$$

where

$$a_c = \frac{\theta}{c+\theta} \quad \text{and} \quad \sigma_c^2 = \frac{c^2}{2\theta^3} \tag{3.10}$$

$$J(c) = -\frac{1}{2} \log\left(\frac{(\theta-c)(\theta+c)}{4c^2}\right). \tag{3.11}$$

(d) *For $c = -\theta$, there exists a sequence (f_k) such that, for any $p > 0$ and T large enough,*

$$\mathbb{P}(\widehat{\theta}_T \leq c) = \frac{\exp(-TI(c))}{2\pi T^{1/4}} \frac{\Gamma(1/4)}{a_\theta^{3/4} \sigma_\theta} \left[1 + \sum_{k=1}^{2p} \frac{f_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p \sqrt{T}}\right)\right] \tag{3.12}$$

where

$$a_\theta = \sqrt{\theta} \quad \text{and} \quad \sigma_\theta^2 = \frac{1}{2\theta}. \tag{3.13}$$

(e) *Finally, for $c = 0$, $p > 0$ and for T large enough,*

$$\mathbb{P}(\widehat{\theta}_T \leq 0) = 2\frac{\exp(-TI(c))\sqrt{2\theta T}}{\sqrt{2\pi}} \times \left[1 + \sum_{k=1}^{p} \frac{(-1)^k (\theta T e^{-2\theta T})^k}{(2k+1)k!} + \mathcal{O}\left((Te^{-2\theta T})^{p+1}\right)\right]. \tag{3.14}$$

Remark 3.4. One can observe that all the sequences $(b_{c,k})$ $(d_{c,k})$, $(e_{c,k})$ may be explicitly calculated as in Theorem 4.1 of [4].

Proof. The proofs are given in Section 4. □

The unstable case $\theta = 0$ can be handled exactly like the explosive case $\theta > 0$ since Lemma 2.1 is also true in the unstable situation. Consequently, we directly obtain the LDP and SLDP for $(\widehat{\theta}_T)$ in the unstable case by replacing θ by 0 in the previous results.

Lemma 3.5. *If $\theta = 0$, then $(\widehat{\theta}_T)$ satisfies an LDP with speed T and good rate function*

$$I(c) = \begin{cases} -\dfrac{c}{4} & \text{if } c \leq 0, \\ 2c & \text{otherwise}. \end{cases} \tag{3.15}$$

Theorem 3.6. *Consider the Ornstein–Uhlenbeck process given by (1.1) where the drift parameter $\theta = 0$.*

(a) *For all $c < 0$, there exists a sequence $(b_{c,k})$ such that, for any $p > 0$ and T large enough,*

$$\mathbb{P}(\widehat{\theta}_T \leq c) = \frac{-2\exp(-TI(c))}{a_c \sigma_c \sqrt{6\pi T}} \left[1 + \sum_{k=1}^{p} \frac{b_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right] \tag{3.16}$$

where $a_c = c/2$ and $\sigma_c^2 = -1/(2c)$.

(b) *For all $c > 0$, there exists a sequence $(d_{c,k})$ such that, for any $p > 0$ and T large enough,*

$$\mathbb{P}(\widehat{\theta}_T \geq c) = \frac{2\exp(-TI(c))}{a_c \sigma_c \sqrt{6\pi T}} \left[1 + \sum_{k=1}^{p} \frac{d_{c,k}}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right) \right] \tag{3.17}$$

where $a_c = 2c$ and $\sigma_c^2 = 1/(16c)$.

4. Proofs of the main results

4.1. Proof of Theorem 3.3(a)

We first focus our attention on the easy case $c < -\theta$. One can observe that a_c, given by (3.4), belongs to the effective domain $\Delta_c =]-\infty, 0[$ whenever $c < -\theta$. Consider the usual change of probability

$$\frac{d\mathbb{P}_T}{d\mathbb{P}} = \exp\left(a_c Z_T(c) - T\mathcal{L}_T(a_c)\right) \tag{4.1}$$

and denote by $\mathbb{E}_T$ the expectation associated with $\mathbb{P}_T$. We clearly have

$$\begin{aligned} \mathbb{P}(\widehat{\theta}_T \leq c) &= \mathbb{P}(Z_T(c) \leq 0) = \mathbb{E}[\mathbb{I}_{Z_T(c) \leq 0}], \\ &= \mathbb{E}_T\left[\exp(-a_c Z_T(c) + T\mathcal{L}_T(a_c))\mathbb{I}_{Z_T(c)\leq 0}\right], \\ &= \exp\left(T\mathcal{L}_T(a_c)\right)\mathbb{E}_T\left[\exp(-a_c Z_T(c))\mathbb{I}_{Z_T(c)\leq 0}\right]. \end{aligned}$$

Consequently, we can split $\mathbb{P}(\widehat{\theta}_T \leq c)$ into two terms: $\mathbb{P}(\widehat{\theta}_T \leq c) = A_T B_T$ with

$$A_T = \exp(T\mathcal{L}_T(a_c)), \tag{4.2}$$

$$B_T = \mathbb{E}_T[\exp(-a_c Z_T(c))\mathbb{I}_{Z_T(c)\leq 0}]. \tag{4.3}$$

On the one hand, we can deduce from (2.2) and (2.5) together with the definition (3.2) of I that

$$A_T = \exp\left(T\mathcal{L}(a_c) + \mathcal{H}(a_c) + \mathcal{R}_T(a_c)\right),$$
$$A_T = \exp\left(-TI(c) + \mathcal{H}(a_c)\right)\left(1 + \mathcal{O}\left(e^{2Tc}\right)\right). \tag{4.4}$$

It only remains to provide the asymptotic expansion of B_T. In all the sequel, the parameter c is omitted in order to simplify the notation in the Taylor expansions.

Lemma 4.1. *For all $c < -\theta$, there exists a sequence (β_k) such that, for any $p > 0$ and T large enough,*

$$B_T = \frac{\beta_0}{\sqrt{T}}\left[1 + \sum_{k=1}^{p}\frac{\beta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]. \tag{4.5}$$

The sequence (β_k) only depends on the derivatives of $\mathcal{L}$ and $\mathcal{H}$ evaluated at point a_c. For example,

$$\beta_0 = -\frac{1}{a_c\sigma_c\sqrt{2\pi}}.$$

Proof. The proof of Lemma 4.1 is given in Appendix C. □

Proof of Theorem 3.3(a). The asymptotic expansion (3.3) immediately follows from (4.4) and (4.5). □

4.2. *Proof of Theorem 3.3(b)*

In the more complicated case $c > \theta$, the effective domain $\Delta_c =]0, 2(c-\theta)[$ and the function $\mathcal{L}$ is decreasing over the interval $]0, 2(c-\theta)[$ as

$$\mathcal{L}'(a) = -\frac{1}{2}\left(1 + \frac{c}{\sqrt{\theta^2 + 2ac}}\right).$$

Consequently, $\mathcal{L}$ reaches its minimum at the value $a_c = 2(c-\theta)$ given by (3.7). Therefore, it is necessary to make use of a slight modification of the strategy of time varying change of probability proposed by Bryc and Dembo [6]. The key point is that there exists a unique a_T, which belongs to the effective domain $\Delta_{T,c}$ of $\mathcal{L}_T$ and converges to a_c as T goes to infinity, which is the solution of the implicit equation given by (4.9) below. Hereafter, we introduce the new probability measure

$$\frac{d\mathbb{P}_T}{d\mathbb{P}} = \exp\left(a_T Z_T(c) - T\mathcal{L}_T(a_T)\right) \tag{4.6}$$

and we denote by $\mathbb{E}_T$ the expectation under $\mathbb{P}_T$. It leads to the decomposition $\mathbb{P}(\widehat{\theta}_T \geq c) = A_T B_T$ where

$$A_T = \exp\left(T\mathcal{L}_T(a_T)\right), \tag{4.7}$$
$$B_T = \mathbb{E}_T\left[\exp(-a_T Z_T(c))\mathbb{I}_{Z_T(c)\geq 0}\right]. \tag{4.8}$$

The proof now splits into two parts: the first one is devoted to the asymptotic expansion of A_T while the second one gives the asymptotic expansion of B_T.

Lemma 4.2. *For all $c > \theta$, there exists a unique a_T, which belongs to the interior of $\Delta_{T,c}$ and converges to $a_c = 2(c - \theta)$ as T goes to infinity, which is the solution of the implicit equation*

$$\mathcal{L}'(a) + \frac{1}{T}\mathcal{H}'(a) = 0 \tag{4.9}$$

where the functions $\mathcal{L}$ and $\mathcal{H}$ are given by (2.1) and (2.3). Moreover, there exists a sequence (γ_k) such that, for any $p > 0$ and T large enough,

$$A_T = \exp\left(-TI(c) + P(c)\right)\sqrt{eT}\left[1 + \sum_{k=1}^{p}\frac{\gamma_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \tag{4.10}$$

where

$$P(c) = -\frac{1}{2}\log\left(\frac{(c-\theta)}{2(2c-\theta)(3c-\theta)}\right). \tag{4.11}$$

The sequence (γ_k) only depends on the Taylor expansion of a_T in the neighborhood of a_c together with the derivatives of $\mathcal{L}$ and $\mathcal{H}$ at point a_c. For example,

$$\gamma_1 = \frac{c(c^2 - 3\theta c + \theta^2)}{2(c-\theta)(\theta - 2c)(3c-\theta)^2}.$$

Proof. The proof of Lemma 4.2 is given in Appendix B. □

It now remains to give the asymptotic expansion of B_T.

Lemma 4.3. *For all $c > \theta$, there exists a sequence (δ_k) such that, for any $p > 0$ and T large enough,*

$$B_T = \sum_{k=1}^{p}\frac{\delta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right). \tag{4.12}$$

The sequence (δ_k) only depends on the Taylor expansion of a_T in the neighborhood of a_c together with the derivatives of $\mathcal{L}$ and $\mathcal{H}$ at point a_c. For example,

$$\delta_1 = \frac{1}{a_c\delta\sqrt{2\pi e}} \quad \text{where } \delta = -\mathcal{L}'(a_c) = \frac{(3c-\theta)}{2(2c-\theta)}.$$

Proof. The proof of Lemma 4.3 is given in Appendix C. □

Proof of Theorem 3.3(b). The asymptotic expansions (4.10) and (4.12) immediately imply (3.6). □

4.3. *Proof of Theorem 3.3(c)*

In the case $|c| < \theta$ and $c \neq 0$, one can easily see that the effective domain is

$$\Delta_c = \begin{cases}]-\infty, 0[& \text{if } -\theta < c < 0, \\ \left]-\dfrac{\theta^2}{2c}, 0\right[& \text{if } 0 < c \leq \dfrac{\theta}{2}, \\ ]2(c-\theta), 0[& \text{if } \dfrac{\theta}{2} \leq c < \theta. \end{cases}$$

In addition, the function $\mathcal{L}$ is always decreasing over Δ_c and $\mathcal{L}$ reaches its minimum at the origin. Consequently, the proof follows essentially the same lines as the one for $c > \theta$ with $a_c = 0$. In fact, with the new probability measure given by (4.6), we have the decomposition $\mathbb{P}(\widehat{\theta}_T \le c) = A_T B_T$ where

$$A_T = \exp\left(T\mathcal{L}_T(a_T)\right), \tag{4.13}$$

$$B_T = \mathbb{E}_T\left[\exp(-a_T Z_T(c))\mathbb{I}_{Z_T(c)\le 0}\right]. \tag{4.14}$$

The proof is also divided into two parts: the first one is devoted to the asymptotic expansion of A_T while the second one gives the asymptotic expansion of B_T.

Lemma 4.4. *For all $|c| < \theta$ and $c \neq 0$, there exists a unique a_T, which belongs to the interior of $\Delta_{T,c}$ and converges to the origin as T goes to infinity, which is the solution of the implicit equation*

$$\mathcal{L}'(a) + \frac{1}{T}\mathcal{H}'(a) = 0 \tag{4.15}$$

where the functions $\mathcal{L}$ and $\mathcal{H}$ are given by (2.1) and (2.3). Moreover, there exists a sequence (γ_k) such that, for any $p > 0$ and T large enough,

$$A_T = \exp\left(-TI(c) + P(c)\right)\sqrt{eT}\left[1 + \sum_{k=1}^{p}\frac{\gamma_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right] \tag{4.16}$$

where

$$P(c) = -\frac{1}{2}\log\left(\frac{(\theta - c)}{2\theta(c+\theta)}\right). \tag{4.17}$$

The sequence (γ_k) only depends on the Taylor expansion of a_T in the neighborhood of the origin together with the derivatives of $\mathcal{L}$ and $\mathcal{H}$ at 0. For example,

$$\gamma_1 = -\frac{c(c^2 + \theta c - \theta^2)}{2\theta(c-\theta)(c+\theta)^2}.$$

Proof. The proof of Lemma 4.4 is given in Appendix B. □

The asymptotic expansion of B_T is as follows.

Lemma 4.5. *For all $|c| < \theta$ and $c \neq 0$, there exists a sequence (δ_k) such that, for any $p > 0$ and T large enough,*

$$B_T = \sum_{k=1}^{p}\frac{\delta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right). \tag{4.18}$$

The sequence (δ_k) only depends on the Taylor expansion of a_T in the neighborhood of the origin together with the derivatives of $\mathcal{L}$ and $\mathcal{H}$ at 0. For example,

$$\delta_1 = \frac{1}{a_c\delta\sqrt{2\pi e}} \quad \text{where } \delta = -\mathcal{L}'(0) = -\frac{(c+\theta)}{2\theta}.$$

Proof. The proof of Lemma 4.5 is given in Appendix C. □

Proof of Theorem 3.3(c). The asymptotic expansion (3.9) immediately follows from (4.16) and (4.18). □

4.4. Proof of Theorem 3.3(d)

In the particular case $c = -\theta$, $\Delta_c =]-\infty, 0[$ and we find a new regime in the asymptotic expansions of a_T, A_T, and B_T.

Lemma 4.6. *For $c = -\theta$, there exists a unique a_T, which belongs to the interior of $\Delta_{T,c}$ and converges to the origin as T goes to infinity, which is the solution of the implicit equation*

$$\mathcal{L}'(a) + \frac{1}{T}\mathcal{H}'(a) = 0 \tag{4.19}$$

where the functions $\mathcal{L}$ and $\mathcal{H}$ are given by (2.1) and (2.3). Moreover, there exists a sequence (γ_k) such that, for any $p > 0$ and T large enough,

$$A_T = \exp\left(-TI(c)\right)(e\theta T)^{1/4}\left[1 + \sum_{k=1}^{2p} \frac{\gamma_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right]. \tag{4.20}$$

The sequence (γ_k) only depends on the Taylor expansion of a_T in the neighborhood of the origin together with the derivatives of $\mathcal{L}$ and $\mathcal{H}$ at 0. For example,

$$\gamma_1 = \frac{3}{8\sqrt{\theta}}.$$

Proof. The proof of Lemma 4.6 is given in Appendix B. □

It now remains to give the asymptotic expansion of B_T.

Lemma 4.7. *For $c = -\theta$, there exists a sequence (δ_k) such that, for any $p > 0$ and T large enough,*

$$B_T = \sum_{k=1}^{2p} \frac{\delta_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right). \tag{4.21}$$

The sequence (δ_k) only depends on the Taylor expansion of a_T in the neighborhood of the origin together with the derivatives of $\mathcal{L}$ and $\mathcal{H}$ at 0. For example,

$$\delta_1 = \frac{1}{2\pi} e^{-1/4} \Gamma\left(\frac{1}{4}\right).$$

Proof. The proof of Lemma 4.7 is given in Appendix C. □

Proof of Theorem 3.3(d). We immediately deduce (3.12) from (4.20) together with (4.21). □

4.5. Proof of Theorem 3.3(e)

We obtain from (3.1) that X_T is Gaussian with an $\mathcal{N}(0, \sigma_T^2)$ distribution where

$$\sigma_T^2 = \frac{1}{2\theta}\left(\exp(2\theta T) - 1\right).$$

Moreover, we clearly have

$$\begin{aligned}\mathbb{P}(\widehat{\theta}_T \leq 0) &= \mathbb{P}(X_T^2 \leq T) = \mathbb{P}(|X_T| \leq \sqrt{T}) = 2\mathbb{P}(0 \leq X_T \leq \sqrt{T}) \\ &= 2\mathbb{P}(0 \leq Z \leq d_T)\end{aligned} \tag{4.22}$$

where Z is an $\mathcal{N}(0,1)$ random variable and the sequence (d_T) satisfies

$$d_T = \sqrt{2\theta T \exp(-2\theta T)}\left[1 + \mathcal{O}(\exp(-2\theta T))\right].$$

For all $x > 0$, define

$$F(x) = \int_0^x f(t)\,dt$$

where f stands for the probability density function of the $\mathcal{N}(0,1)$ distribution. It is well-known that for all $n \geq 1$, the Gaussian derivatives are

$$f^{(n)}(x) = \frac{(-1)^n}{2^{n/2}} H_n\left(\frac{x}{\sqrt{2}}\right) f(x)$$

where (H_n) is the sequence of Hermite polynomials. For example, we have $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = -2 + 4x^2$, etc. Hence,

$$f^{(n)}(0) = \frac{(-1)^n}{\sqrt{2\pi}\,2^{n/2}} H_n$$

where $H_n = H_n(0)$ are the Hermite numbers given by the relation $H_n = -2(n-1)H_{n-2}$ with $H_0 = 1$ and $H_1 = 0$ which implies that

$$H_n = \begin{cases} 0 & \text{if } n \text{ is odd,} \\ \dfrac{(-1)^{n/2} n!}{(n/2)!} & \text{if } n \text{ is even.} \end{cases}$$

Consequently, in the neighborhood of the origin, we have, for all $x > 0$, the Taylor expansion also given in [14, p. 16]:

$$F(x) = \frac{x}{\sqrt{2\pi}}\left[1 + \sum_{k=1}^{p} \frac{(-1)^k x^{2k}}{(2k+1)2^k k!} + \mathcal{O}(x^{p+1})\right]. \tag{4.23}$$

Therefore, we deduce from the identity $\mathbb{P}(\widehat{\theta}_T \leq 0) = 2F(d_T)$ together with (4.23) that

$$\mathbb{P}(\widehat{\theta}_T \leq 0) = 2\frac{\exp(-\theta T)\sqrt{2\theta T}}{\sqrt{2\pi}}\left[1 + \sum_{k=1}^{p} \frac{(-1)^k (\theta T e^{-2\theta T})^k}{(2k+1)k!} + \mathcal{O}\left((Te^{-2\theta T})^{p+1}\right)\right],$$

which immediately leads to (3.14). □

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Appendix A. Proof of the keystone lemma, Lemma 2.1

Our goal is to prove the asymptotic expansion (2.2) associated with the normalized cumulant generating function $\mathcal{L}_T$. Via the same approach as in Section 17.3 of Liptser and Shiryaev [15], we have

$$\begin{aligned}\mathcal{L}_T(a) &= \frac{1}{T}\log\mathbb{E}\left[\exp\left(a\int_0^T X_t dX_t - ac\int_0^T X_t^2\,dt\right)\right],\\ &= \frac{1}{T}\log\mathbb{E}_\varphi\left[\exp\left((a+\theta-\varphi)\int_0^T X_t dX_t + \frac{1}{2}(-2ac-\theta^2+\varphi^2)\int_0^T X_t^2\,dt\right)\right]\end{aligned}$$

for all $\varphi \in \mathbb{R}$, where $\mathbb{E}_\varphi$ stands for the expectation after the change of probability measures

$$\frac{d\mathbb{P}_\varphi}{d\mathbb{P}} = \exp\left((\varphi-\theta)\int_0^T X_t dX_t - \frac{1}{2}(\varphi^2-\theta^2)\int_0^T X_t^2\,dt\right).$$

Hereafter, consider $a \in \Delta_c = \{a \in \mathbb{R}, \theta^2+2ac > 0, a+\theta < \sqrt{\theta^2+2ac}\}$ so that we can choose $\varphi = \varphi(a)$ where $\varphi(a) = -\sqrt{\theta^2+2ac}$. Then, if we define $\tau(a) = a+\theta-\varphi(a)$, we obtain that

$$\mathcal{L}_T(a) = \frac{1}{T}\log\mathbb{E}_\varphi\left[\exp\left(\tau(a)\int_0^T X_t dX_t\right)\right]. \tag{A.1}$$

However, we have from Itô's formula that

$$\int_0^T X_t dX_t = \frac{1}{2}\left(X_T^2 - T\right).$$

Consequently, we obtain from (A.1) that

$$\mathcal{L}_T(a) = -\frac{\tau(a)}{2} + \frac{1}{T}\log\mathbb{E}_\varphi\left[\exp\left(\frac{\tau(a)}{2}X_T^2\right)\right]. \tag{A.2}$$

Under the measure $\mathbb{P}_\varphi$, X_T is a Gaussian random variable with zero mean and variance $\sigma_T^2(a)$ given by

$$\sigma_T^2(a) = -\frac{1-\exp(2\varphi(a)T)}{2\varphi(a)}.$$

Hence, it follows from (A.2) that

$$\mathcal{L}_T(a) = -\frac{\tau(a)}{2} - \frac{1}{2T}\log\left(1 + \frac{\tau(a)}{2\varphi(a)}\left(1-\exp(2\varphi(a)T)\right)\right). \tag{A.3}$$

Finally, if

$$h(a) = \frac{a+\theta}{\varphi(a)} = \frac{\tau(a)}{\varphi(a)} + 1,$$

we find from (A.3) the decomposition

$$\begin{aligned}\mathcal{L}_T(a) &= -\frac{\tau(a)}{2} - \frac{1}{2T}\log\left(1 + \frac{1}{2}(h(a)-1)\left(1-\exp(2\varphi(a)T)\right)\right), \\ &= -\frac{\tau(a)}{2} - \frac{1}{2T}\log\left(\frac{1}{2}(1+h(a)) + \frac{1}{2}(1-h(a))\exp(2\varphi(a)T)\right), \\ &= -\frac{\tau(a)}{2} - \frac{1}{2T}\log\left(\frac{1}{2}(1+h(a))\right) - \frac{1}{2T}\log\left(1 + \frac{1-h(a)}{1+h(a)}\exp(2\varphi(a)T)\right), \\ &= \mathcal{L}(a) + \frac{1}{T}\mathcal{H}(a) + \frac{1}{T}\mathcal{R}_T(a).\end{aligned}$$

One can observe that the remainder $\mathcal{R}_T(a)$ goes to zero exponentially fast as $\mathcal{R}_T(a) = \mathcal{O}(\exp(2\varphi(a)T))$, which completes the proof of Lemma 2.1. □

Appendix B. On the expansions of A_T

All asymptotic expansions associated with A_T are related to the fact that there exists a unique a_T, which belongs to the effective domain $\Delta_{T,c}$ of $\mathcal{L}_T$ and converges to $a_c = 2(c-\theta)$ if $c > \theta$, and to the origin if $|c| < \theta$, which is the solution of the implicit equation

$$\mathcal{L}'(a) + \frac{1}{T}\mathcal{H}'(a) = 0 \tag{B.1}$$

where the functions $\mathcal{L}$ and $\mathcal{H}$ are given by (2.1) and (2.3). After some straightforward calculation, (B.1) can be rewritten as

$$T\varphi(a)(\varphi(a)-c)(\varphi(a)+a+\theta) = c(a+\theta) - \varphi^2(a). \tag{B.2}$$

One can observe that (B.2) may also be rewritten as

$$T\varphi(a)(\varphi(a)-c)(\varphi(a)+\theta)(\varphi(a)+2c-\theta) = -\frac{c}{2}(\varphi^2(a)+\theta^2-2\theta c)$$

which ensures that $\varphi(a_T)$ converges to $\theta - 2c$, while $a_T < 2(c-\theta)$ and a_T converges to a_c. Moreover, it follows from (2.2) that

$$\begin{aligned}A_T &= \exp\left(T\mathcal{L}(a_T) + \mathcal{H}(a_T) + \mathcal{R}_T(a_T)\right), \\ &= \exp\left(T\mathcal{L}(a_T)\right)\exp\left(\mathcal{H}(a_T)\right)\exp\left(\mathcal{R}_T(a_T)\right).\end{aligned} \tag{B.3}$$

Therefore, the proofs of the expansions of A_T are divided into four steps. The first one is devoted to the asymptotic expansions of a_T and $\varphi(a_T)$. The last three deal with the asymptotic expansions of all terms in (B.3).

B.1. *Proof of Lemma 4.2*

Step 1. *One can find two sequences* (a_k) *and* (φ_k) *such that, for any* $p > 0$ *and* T *large enough,*

$$a_T = \sum_{k=0}^{p}\frac{a_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

where $a_0 = 2(c-\theta)$,

$$a_1 = \frac{\theta - 2c}{3c-\theta} \quad \text{and} \quad a_2 = -\frac{c(c^2 - 5\theta c + 2\theta^2)}{2(c-\theta)(3c-\theta)^3},$$

$$\varphi(a_T) = \sum_{k=0}^{p} \frac{\varphi_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

where $\varphi_0 = \theta - 2c$,

$$\varphi_1 = \frac{c}{3c-\theta} \quad \text{and} \quad \varphi_2 = \frac{c^2(4c^2 - 9\theta c + 3\theta^2)}{2(c-\theta)(2c-\theta)(3c-\theta)^3}.$$

Proof. We are in the situation where $c > \theta$, $a_c = 2(c-\theta)$ and $\varphi(a_c) = \theta - 2c$. Consequently, $\varphi(a_c) - c = \theta - 3c \neq 0$ while $\varphi(a_c) + a_c + \theta = 0$. One can easily deduce from (B.2) that

$$\lim_{T\to\infty} T(\varphi(a_T) + a_T + \theta) = \frac{c-\theta}{\theta - 3c}. \tag{B.4}$$

Therefore, the conjunction of (B.2) and (B.4) leads to the asymptotic expansions of a_T and $\varphi(a_T)$. Let us explain now how to calculate the first terms of the expansions. On the one hand, as

$$\varphi(a_T) = -\sqrt{\theta^2 + 2a_T c},$$

we have

$$a_0 = \frac{\varphi_0^2 - \theta^2}{2c}, \qquad a_1 = \frac{\varphi_0\varphi_1}{c}, \qquad a_2 = \frac{2\varphi_0\varphi_2 + \varphi_1^2}{2c}. \tag{B.5}$$

On the other hand, it follows from (B.2) that

$$\varphi_1 + a_1 = \frac{c(a_0+\theta) - \varphi_0^2}{\varphi_0(\varphi_0 - c)},$$

$$\varphi_2 + a_2 = \frac{ca_1 - 2\varphi_0\varphi_1 - (\varphi_1 + a_1)\varphi_1(2\varphi_0 - c)}{\varphi_0(\varphi_0 - c)}.$$

Finally, in order to calculate a_1, a_2, φ_1, and φ_2, it is only necessary to solve very simple linear systems. The rest of the proof is left to the reader. □

Step 2. *One can find a sequence* (α_k) *such that, for any* $p > 0$ *and* T *large enough*,

$$\exp\left(T\mathcal{L}(a_T)\right) = \exp\left(-TI(c) + \frac{1}{2}\right)\left[1 + \sum_{k=1}^{p} \frac{\alpha_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]. \tag{B.6}$$

The sequence (α_k) *only depends on* (a_k) *together with the derivatives of* $\mathcal{L}$ *at point* a_c. *For example*,

$$\alpha_1 = \frac{c(c^2 - 3\theta c + \theta^2)}{2(c-\theta)(2c-\theta)(3c-\theta)^2}.$$

Proof. By the Taylor expansion of $\mathcal{L}$ in the neighborhood of a_c, we have the existence of a sequence (ℓ_k) such that, for any $p > 0$ and T large enough,

$$T\mathcal{L}(a_T) = T\mathcal{L}(a_c) + a_1\mathcal{L}'(a_c) + \sum_{k=1}^{p} \frac{\ell_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right). \tag{B.7}$$

On the one hand,

$$a_1\mathcal{L}'(a_c) = \frac{1}{2}.$$

On the other hand,

$$\ell_1 = a_2\mathcal{L}'(a_c) + \frac{1}{2}a_1^2\mathcal{L}''(a_c) = \frac{c(c^2 - 3\theta c + \theta^2)}{2(c-\theta)(2c-\theta)(3c-\theta)^2}.$$

Therefore, (B.6) clearly follows from (B.7). □

Step 3. *One can find a sequence* (β_k) *such that, for any* $p > 0$ *and* T *large enough,*

$$\exp\left(\mathcal{H}(a_T)\right) = \sqrt{\frac{2\varphi_0 T}{\varphi_1 + a_1}}\left[1 + \sum_{k=1}^{p}\frac{\beta_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right]. \tag{B.8}$$

The sequence (β_k) *only depends on* (a_k) *together with the derivatives of* $\mathcal{H}$ *at point* a_c. *For example,*

$$\beta_1 = \frac{c(c^2 - 3\theta c + \theta^2)}{(c-\theta)(\theta-2c)(3c-\theta)^2}.$$

Proof. By the very definition of $\mathcal{H}$, we have

$$\exp\left(\mathcal{H}(a_T)\right) = \sqrt{\frac{2\varphi(a_T)T}{T(\varphi(a_T) + a_T + \theta)}}.$$

Consequently, the expansion of the square root, together with those of a_T and $\varphi(a_T)$, ensure the existence of a sequence (β_k) such that (B.8) is true. Moreover, as for (α_k), the sequence (β_k) can be explicitly calculated. For example

$$\beta_1 = \frac{1}{2}\left(\frac{\varphi_1}{\varphi_0} - \frac{\varphi_2 + a_2}{\varphi_1 + a_1}\right) = \frac{c(c^2 - 3\theta c + \theta^2)}{(c-\theta)(\theta-2c)(3c-\theta)^2}. \quad \square$$

Step 4. *The remainder* $\mathcal{R}_T(a_T)$ *goes to zero exponentially fast:*

$$\mathcal{R}_T(a_T) = \mathcal{O}\left(T\exp(2\varphi_0 T)\right). \tag{B.9}$$

Proof. The result follows from (2.4) together with the fact that $\varphi_0 < -\theta < 0$. More precisely, we have

$$\frac{1-h(a_T)}{1+h(a_T)} = \frac{T(\varphi(a_T) - a_T - \theta)}{T(\varphi(a_T) + a_T + \theta)}$$

which implies via (B.4) that

$$\lim_{T\to\infty}\frac{1}{T}\left(\frac{1-h(a_T)}{1+h(a_T)}\right) = \frac{\varphi_0 - a_0 - \theta}{\varphi_1 + a_1} = \frac{2(2c-\theta)(3c-\theta)}{c-\theta}. \tag{B.10}$$

Consequently, we immediately deduce (B.9) from (2.4) and (B.10). □

Proof of Lemma 4.2. It follows from the conjunction of (B.3), (B.6), (B.8) and (B.9) that there exists a sequence (γ_k) such that, for any $p > 0$ and T large enough,

$$A_T = \exp\left(-TI(c) + \frac{1}{2}\right)\sqrt{\frac{2\varphi_0 T}{\varphi_1 + a_1}}\left[1 + \sum_{k=1}^{p}\frac{\gamma_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right],$$
$$= \exp\left(-TI(c) + P(c)\right)\sqrt{eT}\left[1 + \sum_{k=1}^{p}\frac{\gamma_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)\right], \tag{B.11}$$

where $P(c)$ is given by (4.11). Finally, the sequence (γ_k) can be explicitly calculated by the use of (a_k) together with the derivatives of $\mathcal{L}$ and $\mathcal{H}$ at point a_c. For example,

$$\gamma_1 = \alpha_1 + \beta_1 = \frac{c(c^2 - 3\theta c + \theta^2)}{2(c-\theta)(\theta - 2c)(3c-\theta)^2}. \quad \square$$

B.2. Proof of Lemma 4.4

We are in the situation where $|c| < \theta$ and $c \neq 0$ which means that $a_c = 0$ and $\varphi(a_c) = -\theta$. Consequently, $\varphi(a_c) - c = -(\theta + c) \neq 0$ while $\varphi(a_c) + a_c + \theta = 0$. The proof of Lemma 4.4 follows exactly the same lines as those of Lemma 4.2. The only notable point to mention is that

$$a_T = \sum_{k=0}^{p}\frac{a_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

where $a_0 = 0$,

$$a_1 = -\frac{\theta}{c+\theta} \quad \text{and} \quad a_2 = -\frac{c(c^2 + 3\theta c - 2\theta^2)}{2(c-\theta)(c+\theta)^3},$$
$$\varphi(a_T) = \sum_{k=0}^{p}\frac{\varphi_k}{T^k} + \mathcal{O}\left(\frac{1}{T^{p+1}}\right)$$

where $\varphi_0 = -\theta$,

$$\varphi_1 = \frac{c}{c+\theta} \quad \text{and} \quad \varphi_2 = \frac{c^2(2c^2 + 3\theta c - 3\theta^2)}{2\theta(c-\theta)(c+\theta)^3}.$$

Therefore, the rest of the proof of the Lemma 4.4 is left to the reader. $\square$

B.3. Proof of Lemma 4.6

The proof of Lemma 4.6 is slightly different from that of Lemma 4.2. More precisely, there is a change of regime in the asymptotic expansions of a_T and $\varphi(a_T)$.

Step 1. *One can find two sequences (a_k) and (φ_k) such that, for any $p > 0$ and T large enough,*

$$a_T = \sum_{k=0}^{2p}\frac{a_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)$$

where $a_0 = 0$, $a_1 = -\sqrt{\theta}$, and $a_2 = -1/8$,

$$\varphi(a_T) = \sum_{k=0}^{2p}\frac{\varphi_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)$$

where $\varphi_0 = -\theta$, $\varphi_1 = -\sqrt{\theta}$, and $\varphi_2 = 3/8$.

Proof. We are in the situation where $c = -\theta, a_c = 0$ and $\varphi(a_c) = -\theta$ which clearly implies that $\varphi(a_c) - c = -(\theta + c) = 0$ and $\varphi(a_c) + a_c + \theta = 0$. It leads to a change of regime in the expansions of a_T and $\varphi(a_T)$ comparing to the expansions of a_T and $\varphi(a_T)$ in Lemma 4.2. As a matter of fact, we obtain from (B.2) that

$$\lim_{T\to\infty} T(\varphi(a_T) + \theta)(\varphi(a_T) + a_T + \theta) = 2\theta. \tag{B.12}$$

Therefore, one can easily deduce the expansions of a_T and $\varphi(a_T)$ from (B.2) and (B.12). The calculation of the first terms is straightforward. For example, as

$$\varphi(a_T) = -\sqrt{\theta^2 - 2\theta a_T},$$

we obtain that $a_0 = 0$, $\varphi_0 = -\theta$,

$$\varphi_1 = a_1 \quad \text{and} \quad \varphi_2 = a_2 + \frac{1}{2}.$$

In addition, we infer from (B.2) that

$$\varphi_1(a_1 + \varphi_1) = 2\theta \quad \text{and} \quad a_2 + 3\varphi_2 = 1.$$

Consequently, we immediately obtain that $a_1^2 = \theta$ which implies that $a_1 = -\sqrt{\theta}$ as a_T belongs to the interior of $\Delta_c =]-\infty, 0[$. It remains to solve the simple linear system

$$\begin{cases} 2a_2 - 2\varphi_2 = -1, \\ a_2 + 3\varphi_2 = 1 \end{cases}$$

whose solution is $a_2 = -1/8$ and $\varphi_2 = 3/8$. □

Step 2. *One can find a sequence* (α_k) *such that, for any* $p > 0$ *and* T *large enough,*

$$\exp(T\mathcal{L}(a_T)) = \exp\left(-TI(c) + \frac{1}{4}\right)\left[1 + \sum_{k=1}^{2p} \frac{\alpha_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right]. \tag{B.13}$$

The sequence (α_k) *only depends on* (a_k) *together with the derivatives of* $\mathcal{L}$ *at the origin. For example,*

$$\alpha_1 = -\frac{3}{16\sqrt{\theta}}.$$

Proof. By the Taylor expansion of $\mathcal{L}$ in the neighborhood of the origin, as $\mathcal{L}'(0) = 0$, we have the existence of a sequence (ℓ_k) such that, for any $p > 0$ and T large enough,

$$T\mathcal{L}(a_T) = T\mathcal{L}(0) + \frac{a_1^2}{2}\mathcal{L}^{(2)}(0) + \sum_{k=1}^{2p} \frac{\ell_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right). \tag{B.14}$$

On the one hand, $\mathcal{L}^{(2)}(0) = 1/(2\theta)$ which implies that

$$\frac{a_1^2}{2}\mathcal{L}^{(2)}(0) = \frac{1}{4}.$$

On the other hand, as $\mathcal{L}^{(3)}(0) = 3/(2\theta^2)$, we also have

$$\ell_1 = a_1 a_2 \mathcal{L}^{(2)}(0) + \frac{a_1^3}{6}\mathcal{L}^{(3)}(0) = -\frac{3}{16\sqrt{\theta}}.$$

Therefore, we deduce (B.13) from (B.14). $\square$

Step 3. *One can find a sequence* (β_k) *such that, for any* $p > 0$ *and* T *large enough,*

$$\exp(\mathcal{H}(a_T)) = (\theta T)^{1/4}\left[1 + \sum_{k=1}^{2p} \frac{\beta_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right]. \tag{B.15}$$

The sequence (β_k) *only depends on* (a_k) *together with the derivatives of* $\mathcal{H}$ *at the origin. For example,*

$$\beta_1 = \frac{9}{16\sqrt{\theta}}.$$

Proof. By the very definition of $\mathcal{H}$, we have

$$\exp(\mathcal{H}(a_T)) = \sqrt{\frac{2\varphi(a_T)\sqrt{T}}{\sqrt{T}(\varphi(a_T) + a_T + \theta)}}.$$

Hence, the expansion of the square root, together with those of a_T and $\varphi(a_T)$, ensure the existence of a sequence (β_k) such that (B.15) is true. As before, the sequence (β_k) can be explicitly calculated. For example,

$$\beta_1 = \frac{1}{2}\left(\frac{\varphi_1}{\varphi_0} - \frac{\varphi_2 + a_2}{\varphi_1 + a_1}\right) = \frac{9}{16\sqrt{\theta}}.$$

Step 4. *The remainder* $\mathcal{R}_T(a_T)$ *goes to zero exponentially fast:*

$$\mathcal{R}_T(a_T) = \mathcal{O}\left(\sqrt{T}\exp(-2\theta T)\right). \quad \square \tag{B.16}$$

Proof. We have

$$\frac{1 - h(a_T)}{1 + h(a_T)} = \frac{\sqrt{T}(\varphi(a_T) - a_T - \theta)}{\sqrt{T}(\varphi(a_T) + a_T + \theta)},$$

which implies that

$$\lim_{T\to\infty} \frac{1}{\sqrt{T}}\left(\frac{1 - h(a_T)}{1 + h(a_T)}\right) = \sqrt{\theta}. \tag{B.17}$$

Consequently, we immediately deduce (B.16) from (2.4) and (B.17). $\square$

Proof of Lemma 4.6. It follows from (B.3), together with (B.13), (B.15) and (B.16), that there exists a sequence (γ_k) such that, for any $p > 0$ and T large enough,

$$A_T = \exp(-TI(c))\,(e\theta T)^{1/4}\left[1 + \sum_{k=1}^{2p} \frac{\gamma_k}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{1}{T^p\sqrt{T}}\right)\right]$$

where the sequence (γ_k) can be explicitly calculated by the use of (a_k) together with the derivatives of $\mathcal{L}$ and $\mathcal{H}$ at the origin. For example,

$$\gamma_1 = \alpha_1 + \beta_1 = \frac{3}{8\sqrt{\theta}}. \quad \square$$

Appendix C. On the expansions of B_T

C.1. General considerations

In order to unify the notation, let $\alpha_T = a_c$ if $c < -\theta$ and $\alpha_T = a_T$ otherwise. In addition, define

$$\beta_T = \begin{cases} \sigma_c\sqrt{T} & \text{if } c < -\theta, \\ \sqrt{T} & \text{if } c = -\theta, \\ T & \text{if } |c| < \theta, \\ -T & \text{if } c > \theta. \end{cases}$$

One can observe that we always have $\alpha_T\beta_T < 0$. Then, in all different cases,

$$B_T = \mathbb{E}_T\left[\exp(-\alpha_T\beta_T U_T)\mathbb{I}_{U_T\leq 0}\right] \tag{C.1}$$

where

$$U_T = \frac{Z_T(c)}{\beta_T}.$$

Denote by Φ_T the characteristic function of U_T under $\mathbb{P}_T$ and assume in all the sequel that $c \neq 0$.

Lemma C.1. *For T large enough, Φ_T belongs to $\mathbb{L}^2(\mathbb{R})$ and, for all $u \in \mathbb{R}$,*

$$\Phi_T(u) = \exp\left(T\mathcal{L}_T\left(\alpha_T + \frac{iu}{\beta_T}\right) - T\mathcal{L}_T(\alpha_T)\right). \tag{C.2}$$

Moreover, we can split B_T into two terms: $B_T = C_T + D_T$ where

$$C_T = -\frac{1}{2\pi\alpha_T\beta_T}\int_{|u|\leq s_T}\left(1 + \frac{iu}{\alpha_T\beta_T}\right)^{-1}\Phi_T(u)du, \tag{C.3}$$

$$D_T = -\frac{1}{2\pi\alpha_T\beta_T}\int_{|u|> s_T}\left(1 + \frac{iu}{\alpha_T\beta_T}\right)^{-1}\Phi_T(u)du. \tag{C.4}$$

where s_T is chosen in such a way that there are positive constants C and $0 < \nu < 1$ satisfying

$$\min\left(\frac{Ts_T^2}{\beta_T^2}, \frac{T\sqrt{s_T}}{\sqrt{|\beta_T|}}\right) \geq CT^{\nu} \tag{C.5}$$

and there exist two positive constants d and D such that

$$|D_T| \leq dT\exp(-DT^{\nu}). \tag{C.6}$$

We choose s_T large enough to satisfy (C.5) and small enough to enable us to permute the integral and summation in (C.3). The expansion of C_T thus follows from that of Φ_T and some tedious calculations. Finally, (C.6) tells us that the expansion of B_T is nothing but that of C_T.

Proof of Lemma C.1. For all $u \in \mathbb{R}$, we have

$$\begin{aligned}
\Phi_T(u) &= \mathbb{E}_T\left[\exp(iuU_T)\right], \\
&= \mathbb{E}\left[\exp\left(iu\frac{Z_T(c)}{\beta_T}\right)\exp\left(\alpha_T Z_T(c) - T\mathcal{L}_T(\alpha_T)\right)\right], \\
&= \mathbb{E}\left[\exp\left(\left(\alpha_T + \frac{iu}{\beta_T}\right)Z_T(c)\right)\right]\exp\left(-T\mathcal{L}_T(\alpha_T)\right), \\
&= \exp\left(T\mathcal{L}_T\left(\alpha_T + \frac{iu}{\beta_T}\right) - T\mathcal{L}_T(\alpha_T)\right).
\end{aligned}$$

We shall see in Appendix D that for T large enough, $\Phi_T \in \mathbb{L}^2(\mathbb{R})$. Then, it follows from the Parseval formula that

$$\begin{aligned}
B_T &= \mathbb{E}_T\left[\exp(-\alpha_T\beta_T U_T)\mathbb{I}_{U_T\le 0}\right], \\
&= -\frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{\alpha_T\beta_T + iu}\Phi_T(u)du, \\
&= -\frac{1}{2\pi\alpha_T\beta_T}\int_{\mathbb{R}}\left(1 + \frac{iu}{\alpha_T\beta_T}\right)^{-1}\Phi_T(u)du
\end{aligned}$$

which implies that $B_T = C_T + D_T$ where C_T and D_T are given by (C.3) and (C.4). It remains to show that D_T goes exponentially fast to zero. We deduce from the Cauchy–Schwarz inequality that

$$|D_T|^2 \le \frac{1}{4\pi^2\alpha_T^2\beta_T^2}\int_{|u|>s_T}\left(1 + \frac{u^2}{(\alpha_T\beta_T)^2}\right)^{-1}du\int_{|u|>s_T}|\Phi_T(u)|^2\,du. \tag{C.7}$$

On the one hand,

$$\int_{|u|>s_T}\left(1 + \frac{u^2}{(\alpha_T\beta_T)^2}\right)^{-1}du \le |\alpha_T\beta_T|\int_{\mathbb{R}}\frac{1}{1+v^2}\,dv \le |\alpha_T\beta_T|\pi. \tag{C.8}$$

On the other hand, we deduce from (C.2) together with inequality (D.1) that for T large enough,

$$|\Phi_T(u)|^2 \le 4\ell(\alpha_T, c, \theta)\left(1 + \gamma_T^2u^2\right)^{1/4}\exp\left(\frac{T\varphi_T}{8}\gamma_T^2u^2\left(1+\gamma_T^2u^2\right)^{-3/4}\right)$$

where $\varphi_T = \varphi(\alpha_T)$ and

$$\gamma_T = \frac{2|c|}{|\beta_T|\varphi^2(\alpha_T)}.$$

It is not hard to see that we can find a positive constant C_ℓ such that, for T large enough, $\ell(\alpha_T, c, \theta) \le C_\ell T$. Consequently, if $\delta_T = \gamma_T s_T$, we obtain that

$$\begin{aligned}
\int_{|u|>s_T}|\Phi_T(u)|^2\,du &\le 8C_\ell T\int_{s_T}^{\infty}\left(1+\gamma_T^2u^2\right)^{1/4} \\
&\qquad \times \exp\left(\frac{T\varphi_T}{8}\gamma_T^2u^2\left(1+\gamma_T^2u^2\right)^{-3/4}\right)du, \\
&\le \frac{8C_\ell T}{\gamma_T}\int_{\delta_T}^{\infty}\left(1+v^2\right)^{1/4}\exp\left(\frac{T\varphi_T}{8}v^2\left(1+v^2\right)^{-3/4}\right)dv.
\end{aligned} \tag{C.9}$$

Let g and h be the two functions defined on $\mathbb{R}^+$ by

$$g(v) = \frac{v^2}{(1+v^2)^{3/4}} \quad \text{and} \quad h(v) = \frac{v^{3/2}}{(1+v^2)^{3/4}}.$$

One can observe that g and h are both increasing functions on $\mathbb{R}^+$. Moreover, as soon as $v > \delta_T$, $g(v) = \sqrt{v}h(v) > \sqrt{v}h(\delta_T)$. In addition, for all $v \in \mathbb{R}^+$, we also have

$$2^{3/4} g(v) \geq \min\left(v^2, \sqrt{v}\right).$$

Therefore, we obtain from (C.9) that

$$\begin{aligned}\int_{|u|>s_T} |\Phi_T(u)|^2 \, du \leq \frac{8C_\ell T}{\gamma_T} \exp\left(\frac{T\varphi_T}{16} g(\delta_T)\right) \\ \times \int_{\delta_T}^{\infty} 2^{1/4} \max(1, \sqrt{v}) \exp(e_T \sqrt{v}) \, dv\end{aligned} \tag{C.10}$$

where

$$e_T = \frac{T\varphi_T}{16} h(\delta_T).$$

The fact that $\varphi_T < 0$ leads to

$$\begin{aligned}\frac{T\varphi_T}{8} g(\delta_T) &\leq \frac{T\varphi_T}{16} 2^{3/4} g(\delta_T), \\ &\leq \frac{T\varphi_T}{16} \min\left(\delta_T^2, \sqrt{\delta_T}\right), \\ &\leq \frac{T\varphi_T}{16} \min\left(\frac{4c^2}{\varphi_T^4}\frac{s_T^2}{\beta_T^2}, \sqrt{\frac{2|c|}{\varphi_T^2}}\sqrt{\frac{s_T}{|\beta_T|}}\right), \\ &\leq \max\left(\frac{c^2}{4\varphi_T^3}, -\frac{\sqrt{2|c|}}{16}\right) \min\left(T\frac{s_T^2}{\beta_T^2}, T\sqrt{\frac{s_T}{|\beta_T|}}\right), \\ &\leq -\mu C T^\nu\end{aligned}$$

where the positive constant μ in the last inequality is due to the boundedness of the terms in the max and the power T^ν follows from assumption (C.5). Furthermore, for T large enough, the integral in (C.10) is bounded by 1. As a matter of fact, we have via straightforward calculation on the Gamma function that

$$\int_0^\infty \max(1, \sqrt{v}) \exp(e_T \sqrt{v}) \, dv \leq \frac{1}{e_T} \max\left(1, -\frac{2}{e_T}\right).$$

It is not hard to see from assumption (C.5) that e_T goes to $-\infty$ as T tends to infinity, which clearly implies that this integral is as small as one wishes. Then, we infer from (C.10) that for T large enough,

$$\int_{|u|>s_T} |\Phi_T(u)|^2 \, du \leq \frac{8C_\ell T}{\gamma_T} \exp(-\mu C T^\nu) \leq \frac{8C_\ell T|\beta_T|\varphi_T^2}{2|c|} \exp(-\mu C T^\nu). \tag{C.11}$$

Finally, we deduce from (C.7), (C.8) and (C.11) that for T large enough,

$$|D_T|^2 \leq \frac{8C_\ell T\varphi_T^2}{8\pi|\alpha_T c|} \exp(-\mu C T^\nu)$$

which clearly implies that, for two positive constants d and D,

$$|D_T| \le dT \exp(-DT^{\nu})$$

and completes the proof of Lemma C.1. $\square$

C.2. Proof of Lemma 4.1

Lemma C.2. *For $c < -\theta$, the distribution of U_T under $\mathbb{P}_T$ converges, as T goes to infinity, to an $\mathcal{N}(0,1)$ distribution which means that Φ_T converges to Φ given by*

$$\Phi(u) = \exp\left(-\frac{u^2}{2}\right).$$

Moreover, for any $p > 0$, there exist integers $q(p), r(p)$ and a sequence $(\varphi_{k,l})$ independent of p such that, for T large enough,

$$\Phi_T(u) = \Phi(u)\left[1 + \frac{1}{\sqrt{T}} \sum_{k=0}^{2p} \sum_{l=k+1}^{q(p)} \frac{\varphi_{k,l} u^l}{(\sqrt{T})^k} + \mathcal{O}\left(\frac{\max(1, |u|^{r(p)})}{T^{p+1}}\right)\right] \tag{C.12}$$

where σ_c^2 is given by (3.4) and the remainder $\mathcal{O}$ is uniform as soon as $|u| \le sT^{1/6}$ with $s > 0$.

Proof of Lemma C.2. It follows from (2.2) that for all $k \in \mathbb{N}$,

$$\mathcal{L}_T^{(k)}(a_c) = \mathcal{L}^{(k)}(a_c) + \frac{1}{T}\mathcal{H}^{(k)}(a_c) + \frac{1}{T}\mathcal{R}_T^{(k)}(a_c). \tag{C.13}$$

Moreover, it is rather easy to see that for all $k \in \mathbb{N}$,

$$\mathcal{R}_T^{(k)}(a_c) = \mathcal{O}(T^k \exp(2Tc)). \tag{C.14}$$

One can observe that $\mathcal{L}^{(1)}(a_c) = 0$ and $\mathcal{L}^{(2)}(a_c) = \sigma_c^2$ with σ_c^2 given by (3.4). In addition, taking $\beta_T = \sigma_c\sqrt{T}$, we also have

$$T\left(\frac{iu}{\beta_T}\right)^2 \frac{\mathcal{L}^{(2)}(a_c)}{2} = -\frac{u^2}{2}.$$

Hence, by a Taylor expansion, we find from (C.2), (C.13) and (C.14) that for any $p > 0$,

$$\log \Phi_T(u) = -\frac{u^2}{2} + T\sum_{k=3}^{2p+3}\left(\frac{iu}{\sigma_c\sqrt{T}}\right)^k \frac{\mathcal{L}^{(k)}(a_c)}{k!} + \sum_{k=1}^{2p+1}\left(\frac{iu}{\sigma_c\sqrt{T}}\right)^k \frac{\mathcal{H}^{(k)}(a_c)}{k!} + \mathcal{O}\left(\frac{\max(1, u^{2p+4})}{T^{p+1}}\right).$$

Finally, we deduce the asymptotic expansion (C.12) by taking the exponential on both sides, remarking that, as soon as $|u| \le sT^{1/6}$ with $s > 0$, the quantity $u^l/(\sqrt{T})^k$ remains bounded in (C.12). $\square$

Proof of Lemma 4.1. In order to achieve the proof of the Lemma 4.1, let $s_T = sT^{1/6}$ with $s > 0$ and $\beta_T = \sigma_c\sqrt{T}$. As

$$\min\left(\frac{Ts_T^2}{\beta_T^2}, \frac{T\sqrt{s_T}}{\sqrt{|\beta_T|}}\right) \ge CT^{1/3},$$

the assumption (C.5) of Lemma C.1 is clearly satisfied. Consequently, there exist two positive constants d and D such that

$$|D_T| \leq d \exp(-DT^{1/3}).$$

Finally, we obtain (4.5) from (C.3) and (C.12) together with standard calculations on the $\mathcal{N}(0,1)$ distribution. □

C.3. Proof of Lemma 4.3

Lemma C.3. *For $c > \theta$, the distribution of U_T under $\mathbb{P}_T$ converges, as T goes to infinity, to the distribution of $\gamma(N^2-1)$, where N is an $\mathcal{N}(0,1)$ random variable and*

$$\gamma = \mathcal{L}'(2(c-\theta)) = \frac{(3c-\theta)}{2(\theta-2c)},$$

which means that Φ_T converges to Φ given by

$$\Phi(u) = \frac{\exp(-i\gamma u)}{\sqrt{1-2i\gamma u}}.$$

Moreover, for any $p > 0$, there exist integers $q(p), r(p), s(p)$ and a sequence $(\varphi_{k,l,m})$ independent of p such that, for T large enough,

$$\Phi_T(u) = \Phi(u)\exp\left(-\frac{\sigma_c^2 u^2}{2T}\right) \times \left[1 + \sum_{k=1}^{p}\sum_{l=k+1}^{q(p)}\sum_{m=0}^{r(p)} \frac{\varphi_{k,l,m}u^l}{T^k(1-2i\gamma u)^m} + \mathcal{O}\left(\frac{\max(1,|u|^{s(p)})}{T^{p+1}}\right)\right] \tag{C.15}$$

where σ_c^2 is given by (3.7) and the remainder $\mathcal{O}$ is uniform as soon as $|u| \leq sT^{2/3}$ with $s > 0$.

Proof of Lemma C.3. It follows from (2.2) and (C.2) that

$$\Phi_T(u) = \exp\left(T\left(\mathcal{L}\left(a_T + \frac{iu}{\beta_T}\right) - \mathcal{L}(a_T)\right) + \mathcal{H}\left(a_T + \frac{iu}{\beta_T}\right) - \mathcal{H}(a_T) + \mathcal{R}\left(a_T + \frac{iu}{\beta_T}\right) - \mathcal{R}(a_T)\right) \tag{C.16}$$

where $\beta_T = -T$. We shall focus our attention on each term of (C.16). First, by virtue of Lemma 2.1, the term involving the remainder $\mathcal{R}$ does not contribute to the asymptotic expansion of Φ_T. Next, by the very definition (2.1) of $\mathcal{L}$, the first term of (C.16) can be rewritten as

$$T\left(\mathcal{L}\left(a_T + \frac{iu}{\beta_T}\right) - \mathcal{L}(a_T)\right) = -\frac{T}{2}\left(\frac{iu}{\beta_T} - \varphi_T\left(\left(1 + \frac{iub_T}{\beta_T}\right)^{1/2} - 1\right)\right)$$

where $\varphi_T = -\sqrt{\theta^2 + 2a_T c}$ and $b_T = 2c/\varphi_T^2$. Consequently, as b_T/β_T tends to 0, we have for all $p \geq 2$,

$$\exp\left(T\left(\mathcal{L}\left(a_T + \frac{iu}{\beta_T}\right) - \mathcal{L}(a_T)\right)\right) = \exp\left(-\frac{iuT}{2\beta_T} + \frac{T\varphi_T}{2}\sum_{k=1}^{p} l_k\left(\frac{iub_T}{\beta_T}\right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{T^{p+1}}\right)\right)$$

where $l_k = (-1)^{k-1}(2k)!/((2k-1)(2^k k!)^2)$ which leads to

$$\begin{aligned}
&\exp\left(T\left(\mathcal{L}\left(a_T + \frac{iu}{\beta_T}\right) - \mathcal{L}(a_T)\right)\right)\\
&\quad = \exp\left(-iuc_T - \frac{d_T u^2}{2T}\right)\\
&\qquad \times \exp\left(\frac{T\varphi_T}{2}\sum_{k=3}^{p} l_k\left(\frac{iub_T}{\beta_T}\right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{T^{p+1}}\right)\right)
\end{aligned} \tag{C.17}$$

where

$$c_T = \frac{c-\varphi_T}{2\varphi_T} \quad \text{and} \quad d_T = -\frac{\varphi_T b_T^2}{8}.$$

For the second term of (C.16), we also have, by the very definition (2.3) of $\mathcal{H}$,

$$\begin{aligned}
&\exp\left(\mathcal{H}\left(a_T + \frac{iu}{\beta_T}\right) - \mathcal{H}(a_T)\right)\\
&\quad = \left(\frac{\varphi_T + a_T + \theta}{\varphi_T + \left(a_T + iu\beta_T^{-1} + \theta\right)\left(1 + iub_T\beta_T^{-1}\right)^{-1/2}}\right)^{1/2}.
\end{aligned}$$

Hence, we have for all $p \geq 2$,

$$\begin{aligned}
&\exp\left(\mathcal{H}\left(a_T + \frac{iu}{\beta_T}\right) - \mathcal{H}(a_T)\right)\\
&\quad = \frac{1}{\sqrt{f_T(u)}}\left(1 + g_T(u)u^2 + h_T(u)\left(\sum_{k=2}^{p} h_k\left(\frac{iub_T}{-T}\right)^k + \mathcal{O}\left(\frac{|u|^{p+1}}{T^{p+1}}\right)\right)\right)^{-1/2}
\end{aligned} \tag{C.18}$$

where $h_k = (2k)!/(2^k k!)^2$ and

$$\begin{aligned}
f_T(u) &= 1 - \frac{iu}{e_T} + \frac{(a_T+\theta)iub_T}{2e_T},\\
g_T(u) &= \frac{b_T}{2Te_T f_T(u)},\\
h_T(u) &= \frac{T(a_T+\theta) - iu}{e_T f_T(u)},
\end{aligned}$$

with $e_T = T(\varphi_T + a_T + \theta)$. One can easily check that, as T goes to infinity, the limits of b_T, c_T, d_T, and e_T are respectively given by $2c/(\theta-2c)^2, \gamma, \sigma_c^2$, and $(\theta-c)/(3c-\theta)$ which implies that $f_T(u)$ converges to $1 - 2i\gamma u$. Finally, we find via (C.17) and (C.18) the pointwise convergence

$$\lim_{T\to\infty} \Phi_T(u) = \Phi(u) = \frac{\exp(-i\gamma u)}{\sqrt{1-2i\gamma u}}$$

while (C.15) follows from the Taylor expansion of the exponential in (C.17) together with the Taylor expansion of the square root in (C.18). □

Proof of Lemma 4.3. In order to complete the proof of the Lemma 4.3, let $s_T = sT^{2/3}$ with $s > 0$ and $\beta_T = -T$. It is not hard to see that

$$\min\left(\frac{Ts_T^2}{\beta_T^2}, \frac{T\sqrt{s_T}}{\sqrt{|\beta_T|}}\right) \geq CT^{1/3},$$

which means that the assumption (C.5) of Lemma C.1 is satisfied. Therefore, there exist two positive constants d and D such that

$$|D_T| \leq dT\exp(-DT^{1/3}).$$

Finally, we deduce (4.12) from (C.3) and (C.15) via a careful use of the contour integral lemma for the Gamma function given in Lemma 7.3 of [4]. □

C.4. Proof of Lemma 4.5

Lemma C.4. *For $|c| < \theta$ with $c \neq 0$, the distribution of U_T under $\mathbb{P}_T$ converges, as T goes to infinity, to the distribution of $\gamma(N^2 - 1)$, where N is an $\mathcal{N}(0, 1)$ random variable and*

$$\gamma = -\mathcal{L}'(0) = \frac{(\theta + c)}{2\theta},$$

which means that Φ_T converges to Φ given by

$$\Phi(u) = \frac{\exp(-i\gamma u)}{\sqrt{1 - 2i\gamma u}}.$$

Moreover, for any $p > 0$, there exist integers $q(p), r(p), s(p)$ and a sequence $(\varphi_{k,l,m})$ independent of p, such that, for T large enough,

$$\Phi_T(u) = \Phi(u)\exp\left(-\frac{\sigma_c^2 u^2}{2T}\right) \times \left[1 + \sum_{k=1}^{p}\sum_{l=k+1}^{q(p)}\sum_{m=0}^{r(p)} \frac{\varphi_{k,l,m} u^l}{T^k(1 - 2i\gamma u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right)\right] \tag{C.19}$$

where σ_c^2 is given by (3.10) and the remainder $\mathcal{O}$ is uniform as soon as $|u| \leq sT^{2/3}$ with $s > 0$.

Proof of Lemma C.4. The proof is left to the reader, as it follows essentially the same lines as the proof of Lemma C.3. □

Proof of Lemma 4.5. The proof of Lemma 4.5 follows exactly the same arguments as the proof of Lemma 4.3. The only notable point to mention is that we have to take into account twice the asymptotic expansion of a_T because a_T goes to zero as T tends to infinity and a_T is also in the denominator of C_T. □

C.5. Proof of Lemma 4.7

Lemma C.5. *For $c = -\theta$, the distribution of U_T under $\mathbb{P}_T$ converges, as T goes to infinity, to the distribution of $\sigma_\theta N + \gamma_\theta(M^2 - 1)$, where σ_θ^2 is given by (3.13), N and M are two independent $\mathcal{N}(0, 1)$ random variables and*

$$\gamma_\theta = \frac{1}{2\sqrt{\theta}},$$

which means that Φ_T converges to Φ given by

$$\Phi(u) = \frac{\exp(-i\gamma_\theta u)}{\sqrt{1-2i\gamma_\theta u}} \exp\left(-\frac{\sigma_\theta^2 u^2}{2}\right).$$

Moreover, for any $p > 0$, there exist integers $q(p), r(p), s(p)$ and a sequence $(\varphi_{k,l,m})$ independent of p, such that, for T large enough,

$$\Phi_T(u) = \Phi(u)\left[1 + \frac{1}{\sqrt{T}} \sum_{k=0}^{2p} \sum_{l=k+1}^{q(p)} \sum_{m=0}^{r(p)} \frac{\varphi_{k,l,m} u^l}{(\sqrt{T})^k (1-2i\gamma_\theta u)^m} + \mathcal{O}\left(\frac{\max(1, |u|^{s(p)})}{T^{p+1}}\right)\right] \tag{C.20}$$

where the remainder $\mathcal{O}$ is uniform as soon as $|u| \leq sT^{1/6}$ with $s > 0$.

Proof of Lemma C.5. The proof is left to the reader as it follows essentially the same lines as the proof of Lemma C.3. □

Proof of Lemma 4.7. The proof of Lemma 4.7 follows exactly the same arguments as the proof of Lemma 4.3 with a careful use of the contour integral lemma for the Gamma function given in Lemma 7.3 of [4]. □

Appendix D. Technical lemmas

D.1. Statement of the results

The effective domain $\Delta_{T,c}$ of the normalized cumulant generating function $\mathcal{L}_T$ was previously calculated in Lemma 4.1 of [11]. It is an open interval such that $\widetilde{\Delta}_{T,c} \subset \Delta_{T,c} \subset \overline{\Delta}_{T,c}$ where

$$\widetilde{\Delta}_{T,c} = \left\{a \in \mathbb{R},\ \theta^2 + 2ac > 0,\ a + \theta < \sqrt{\theta^2 + 2ac}\coth(T\sqrt{\theta^2 + 2ac})\right\}$$

and

$$\overline{\Delta}_{T,c} = \left\{a \in \mathbb{R},\ \theta^2 + 2ac + \frac{\pi^2}{T^2} > 0,\ a + \theta < \sqrt{\theta^2 + 2ac}\coth(T\sqrt{\theta^2 + 2ac})\right\}.$$

Define $\mathcal{D}_{T,c} = \{z \in \mathbb{C}, \mathrm{Re}(z) \in \Delta_{T,c}\}$ and $\mathcal{D}_c = \{z \in \mathbb{C}, \mathrm{Re}(z) \in \Delta_c\}$.

Lemma D.1. *For T large enough, Φ_T belongs to $\mathbb{L}^2(\mathbb{R})$. More precisely, for T large enough and for any $(a, u) \in \mathbb{R}^2$ such that $(a + iu) \in \mathcal{D}_{T,c}$,*

$$\left|\exp\left(T\left(\mathcal{L}_T(a+iu) - \mathcal{L}_T(a)\right)\right)\right|^2 \leq 4\ell(a,c,\theta)\left(1 + \frac{4c^2u^2}{\varphi^4(a)}\right)^{1/4} \times \exp\left(T\frac{c^2u^2}{2\varphi^3(a)}\left(1 + \frac{4c^2u^2}{\varphi^4(a)}\right)^{-3/4}\right) \tag{D.1}$$

where

$$\ell(a,c,\theta) = \max\left(1, \frac{|\varphi(a)+\theta|}{|\varphi(a)|}\right)\max\left(1, \frac{|\varphi(a)+2c-\theta|}{|\varphi(a)|}\right). \tag{D.2}$$

D.2. Proof of Lemma D.1

The key point is to make use of a complex counterpart of the main decomposition (2.2), which means that

$$\mathcal{L}_T(z) = \mathcal{L}(z) + \frac{1}{T}\mathcal{H}(z) + \frac{1}{T}\mathcal{R}_T(z) \tag{D.3}$$

where $\mathcal{L}, \mathcal{H}$ and $\mathcal{R}_T$ are respectively given by (2.1), (2.3) and (2.4). In order to make these expressions meaningful, we have to take care with the definitions. We shall denote the principal determination of the logarithm defined on $\mathbb{C}\backslash]-\infty, 0]$ by

$$\log(z) = \log|z| + i\,\mathrm{Arg}(z),$$

where

$$\mathrm{Arg}(z) = \begin{cases} \arcsin\left(\dfrac{\mathrm{Im}(z)}{|z|}\right) & \text{if } \mathrm{Re}(z) \geq 0, \\ \arccos\left(\dfrac{\mathrm{Re}(z)}{|z|}\right) & \text{if } \mathrm{Re}(z) < 0,\ \mathrm{Im}(z) > 0, \\ -\arccos\left(\dfrac{\mathrm{Re}(z)}{|z|}\right) & \text{if } \mathrm{Re}(z) < 0,\ \mathrm{Im}(z) < 0. \end{cases}$$

We also introduce the analytic function defined for all $z \in \mathbb{C}$ with $\mathrm{Re}(1+z) > 0$ by

$$\sqrt{1+z} = \sqrt{|1+z|}\exp\left(\frac{i}{2}\mathrm{Arg}(1+z)\right).$$

It is not hard to see that

$$\mathrm{Re}(\sqrt{1+z}) = \frac{1}{\sqrt{2}}\sqrt{|1+z| + 1 + \mathrm{Re}(z)}. \tag{D.4}$$

The proof of Lemma D.1 follows from the conjunction of three lemmas, each one involving the functions $\mathcal{L}, \mathcal{H}$ and $\mathcal{R}_T$.

Lemma D.2. *The function $\mathcal{L}$ given, for all $z \in \mathbb{C}$, by*

$$\mathcal{L}(z) = -\frac{1}{2}(z + \theta - \varphi(z)) \quad \textit{where } \varphi(z) = -\sqrt{\theta^2 + 2zc}$$

is differentiable on the domain $\mathcal{D}_c$. Moreover, for all $a \in \Delta_c$ and $u \in \mathbb{R}$, we have

$$\left|\exp\left(T(\mathcal{L}(a+iu) - \mathcal{L}(a))\right)\right|^2 \leq \exp\left(T\frac{c^2u^2}{4\varphi^3(a)}\left(1 + \frac{4c^2u^2}{\varphi^4(a)}\right)^{-3/4}\right). \tag{D.5}$$

Proof of Lemma D.2. For all $z \in \mathbb{C}$ such that $\mathrm{Re}(z) \in \Delta_c, \varphi(z)$ is well defined. Hence, φ is differentiable on $\mathcal{D}_c$ and the same is also true for $\mathcal{L}$. In addition, we have

$$\mathcal{L}(a+iu) - \mathcal{L}(a) = -\frac{1}{2}(iu - \varphi(a+iu) + \varphi(a))$$

which clearly implies that

$$\left|\exp\left(T(\mathcal{L}(a+iu) - \mathcal{L}(a))\right)\right| \leq \exp\left(\frac{T}{2}(\mathrm{Re}(\varphi(a+iu) - \varphi(a)))\right). \tag{D.6}$$

Moreover, we also have

$$\varphi(a+iu)-\varphi(a)=\varphi(a)\left(\sqrt{1+\frac{2icu}{\varphi^2(a)}}-1\right).$$

We deduce from (D.4) with $z=2icu/\varphi^2(a)$ that

$$\mathrm{Re}\,(\varphi(a+iu)-\varphi(a))=\frac{\varphi(a)}{\sqrt{2}}\left(\sqrt{\sqrt{1+\frac{4c^2u^2}{\varphi^4(a)}}+1}-\sqrt{2}\right).$$

Keeping in mind that $\varphi(a)<0$, we infer from the elementary inequality

$$\sqrt{\sqrt{1+x}+1}-\sqrt{2}\geq\frac{x}{4\sqrt{2}(1+x)^{3/4}},$$

which is true as soon as $x\geq 0$, that

$$\mathrm{Re}\,(\varphi(a+iu)-\varphi(a))\leq\frac{c^2u^2}{2\varphi^3(a)}\left(1+\frac{4c^2u^2}{\varphi^4(a)}\right)^{-3/4}.$$

Finally, it is ensured via (D.6) that for all $a\in\Delta_c$ and $u\in\mathbb{R}$,

$$\left|\exp\left(T(\mathcal{L}(a+iu)-\mathcal{L}(a))\right)\right|^2\leq\exp\left(T\frac{c^2u^2}{4\varphi^3(a)}\left(1+\frac{4c^2u^2}{\varphi^4(a)}\right)^{-3/4}\right),$$

which completes the proof of Lemma D.2. □

Lemma D.3. *The function $\mathcal{H}$ given, for all $z\in\mathbb{C}$, by*

$$\mathcal{H}(z)=-\frac{1}{2}\log\left(\frac{1}{2}(1+h(z))\right)\quad\text{where } h(z)=\frac{(z+\theta)}{\varphi(z)}$$

is differentiable on the domain $\mathcal{D}_c$. Moreover, for all $a\in\Delta_c$ and $u\in\mathbb{R}$, we have

$$\left|\exp(\mathcal{H}(a+iu)-\mathcal{H}(a))\right|^2\leq\ell(a,c,\theta)\left(1+\frac{4c^2u^2}{\varphi^4(a)}\right)^{1/4}.\tag{D.7}$$

Proof of Lemma D.3. First of all, it follows from (2.3) that

$$\left|\exp\left(\mathcal{H}(a+iu)-\mathcal{H}(a)\right)\right|^2=\left|\frac{1+h(a)}{1+h(a+iu)}\right|.\tag{D.8}$$

We claim that for $z\in\mathbb{C}$ such that $\mathrm{Re}(z)\in\Delta_c$, $1+h(z)\in\mathbb{C}\backslash]-\infty,0]$. Assume by contradiction that this is not true, which means that one can find $\lambda\in[0,+\infty[$ such that

$$1+h(z)=-\lambda.$$

Since, $\varphi^2(z)=\theta^2+2cz$, $\varphi(z)$ is a root of the quadratic equation

$$\varphi^2(z)+2c(1+\lambda)\varphi(z)-\theta^2+2c\theta=0.$$

Its discriminant is non-negative as

$$4(c-\theta)^2+4c^2\lambda^2+8c^2\lambda\geq 0.$$

One can observe that c, θ and λ are real numbers which implies that $\varphi(z)$ is also a real number, as well as z. Consequently, z belongs to Δ_c and $1 + h(z) > 0$ which contradicts the assumption. This allows us to say that $\mathcal{H}$ is differentiable on $\mathcal{D}_c$. We are now in a position to prove inequality (D.7). Since $\varphi^2(z) = \theta^2 + 2cz$, for $z \in \mathbb{C}$ such that $\text{Re}(z) \in \Delta_c$, we have

$$1 + h(z) = \frac{(\varphi(z) + \theta)(\varphi(z) + 2c - \theta)}{2c\varphi(z)}. \tag{D.9}$$

For all $z \in \mathbb{C}$ such that $\text{Re}(z) \in \Delta_c$, and for all $\alpha \in \mathbb{R}$, we clearly have

$$|\varphi(z) + \alpha|^2 = |\varphi(z)|^2 + \alpha^2 + 2\alpha \text{Re}(\varphi(z)).$$

Assume that a belongs to Δ_c and let $u \in \mathbb{R}$. We already saw that

$$\varphi(a + iu) = \varphi(a)\sqrt{1 + \frac{2icu}{\varphi^2(a)}}$$

which leads to

$$|\varphi(a + iu)|^2 = \varphi^2(a)\left|1 + \frac{2icu}{\varphi^2(a)}\right| = \varphi^2(a)\left(1 + \frac{4c^2u^2}{\varphi^4(a)}\right)^{1/4}. \tag{D.10}$$

On the other hand, it also follows from (D.4) that

$$\text{Re}(\varphi(a + iu)) = \frac{\varphi(a)}{\sqrt{2}}\sqrt{1 + \left|1 + \frac{2icu}{\varphi^2(a)}\right|} = \frac{\varphi(a)}{\sqrt{2}}\sqrt{1 + \sqrt{1 + \frac{4c^2u^2}{\varphi^4(a)}}}. \tag{D.11}$$

Consequently, $|\varphi(a + iu)|^2 = 2(\text{Re}(\varphi(a + iu)))^2 - \varphi^2(a)$ which implies that

$$\begin{aligned}\left|\varphi(a + iu) + \alpha\right|^2 = \varphi^2(a)\Bigg(&\frac{2(\text{Re}(\varphi(a + iu)))^2}{\varphi^2(a)} - 1 + \frac{\alpha^2}{\varphi^2(a)} \\ &+ \frac{2\alpha}{\varphi(a)}\frac{\text{Re}(\varphi(a + iu))}{\varphi(a)}\Bigg).\end{aligned} \tag{D.12}$$

By introducing the function

$$g_\alpha(x) = \left(x + \frac{\alpha}{\sqrt{2}}\right)^2 + \frac{\alpha^2}{2} - 1,$$

we deduce from (D.11) together (D.12) with $\beta = \alpha/\varphi(a)$ that

$$\left|\varphi(a + iu) + \alpha\right|^2 = \varphi^2(a) g_\beta\left(\sqrt{1 + \sqrt{1 + \frac{4c^2u^2}{\varphi^4(a)}}}\right). \tag{D.13}$$

Furthermore, one can easily check from straightforward calculations that for all $x \geq \sqrt{2}$,

$$g_\beta(x) \geq \begin{cases} (\beta + 1)^2 & \text{if } \beta \in [-2, 0], \\ 1 & \text{otherwise.} \end{cases} \tag{D.14}$$

Therefore, we infer from (D.13) and (D.14) that for all $a \in \Delta_c$ and for all $u \in \mathbb{R}$,

$$\left|\varphi(a + iu) + \alpha\right| \geq \begin{cases} |\varphi(a)||\beta + 1| & \text{if } \beta \in [-2, 0], \\ |\varphi(a)| & \text{otherwise} \end{cases}$$

which clearly implies that

$$\frac{1}{|\varphi(a+iu)+\alpha|} \leq \frac{1}{|\varphi(a)|} \max\left(1, \frac{|\varphi(a)|}{|\varphi(a)+\alpha|}\right). \tag{D.15}$$

We shall make use of inequality (D.15) with $\alpha = \theta$ and $\alpha = 2c - \theta$. One can observe that, as long as $a \in \Delta_c$, the value of $\varphi(a) + \alpha \neq 0$. Finally, it follows from the conjunction of (D.8), (D.9), (D.10), and (D.15) that for all $a \in \Delta_c$ and for all $u \in \mathbb{R}$,

$$\begin{aligned}\left|\exp\left(\mathcal{H}(a+iu) - \mathcal{H}(a)\right)\right|^2 &\leq \frac{|\varphi(a+iu)|}{|\varphi(a)|} \max\left(1, \frac{|\varphi(a)+\theta|}{|\varphi(a)|}\right) \\ &\quad \times \max\left(1, \frac{|\varphi(a)+2c-\theta|}{|\varphi(a)|}\right), \\ &\leq \ell(a,c,\theta)\left(1+\frac{4c^2u^2}{\varphi^4(a)}\right)^{1/4}\end{aligned}$$

which completes the proof of Lemma D.3. $\square$

Lemma D.4. *For T large enough, the function $\mathcal{R}_T$ given, for all $z \in \mathbb{C}$, by*

$$\mathcal{R}_T(z) = -\frac{1}{2}\log\left(1 + \frac{1-h(z)}{1+h(z)}\exp(2\varphi(z)T)\right)$$

is differentiable on the domain $\mathcal{D}_{T,c}$. Moreover, for all $(a,u) \in \mathbb{R}^2$ such that $a+iu \in \mathcal{D}_{T,c}$, we have

$$\left|\exp(\mathcal{R}_T(a+iu) - \mathcal{R}_T(a))\right|^2 \leq 4. \tag{D.16}$$

Proof of Lemma D.4. First of all, we deduce from (2.4) that

$$\left|\exp\left(\mathcal{R}_T(a+iu) - \mathcal{R}_T(a)\right)\right|^2 = \left|\frac{1+r(a)\exp(2\varphi(a)T)}{1+r(a+iu)\exp(2\varphi(a+iu)T)}\right| \tag{D.17}$$

where the function r is given, for all $z \in \mathbb{C}$, by

$$r(z) = \frac{1-h(z)}{1+h(z)}.$$

We already saw from (D.9) that

$$1 + h(z) = \frac{(\varphi(z)+\theta)(\varphi(z)+2c-\theta)}{2c\varphi(z)}.$$

Hence,

$$1 - h(z) = \frac{(\theta-\varphi(z))(\varphi(z)-2c+\theta)}{2c\varphi(z)}$$

which implies that

$$|r(z)| = \left|\frac{\varphi(z)-\theta}{\varphi(z)+\theta}\right|\left|\frac{\varphi(z)-2c+\theta}{\varphi(z)+2c-\theta}\right|. \tag{D.18}$$

Moreover, for all $(a, u) \in \mathbb{R}^2$ such that $a + iu \in \mathcal{D}_{T,c}$,

$$\left| r(a+iu)\exp(2\varphi(a+iu)T) \right|^2 = |r(a+iu)|^2 \exp(4\mathrm{Re}(\varphi(a+iu))T).$$

We recall from (D.11) that

$$\mathrm{Re}(\varphi(a+iu)) = \frac{\varphi(a)}{\sqrt{2}} \sqrt{1 + \sqrt{1 + \frac{4c^2u^2}{\varphi^4(a)}}}.$$

The key point here is that $\mathrm{Re}(\varphi(a+iu))$ is always negative. Consequently, via the same lines as in the proof of Lemma D.3, we obtain that for T large enough and for all $(a, u) \in \mathbb{R}^2$ such that $a + iu \in \mathcal{D}_{T,c}$,

$$\left| r(a+iu)\exp(2\varphi(a+iu)T) \right| \leq \frac{1}{2}. \tag{D.19}$$

Finally, (D.16) follows from (D.17) and (D.19). □

Proof of Lemma D.1. Lemma D.1 immediately follows from (D.3) together with the conjunction of Lemmas D.2–D.4. □

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